Equilibrium Concepts

Nash Equilibria, Mixed Strategy Equilibria, Coalitional Games, the Core, Shapley Value, Nash Bargaining

Steady-states

- Recall model of \( \langle N, A, \{d\}, T, \rangle \) which we characterize with the evolution function \( d \)
- Steady-state is a point where \( a^* = d(a^*) \) for all \( t \geq t^* \)
- Obvious solution: solve for fixed points of \( d \)
- For non-cooperative radios, if \( a^* \) is a fixed point under synchronous timing, then it is under the other three timings.
- Works well for convex action spaces
  - Not always guaranteed to exist
  - Value of fixed point theorems
- Not so well for finite spaces
  - Generally requires exhaustive search

Nash Equilibrium

“A steady-state where each player holds a correct expectation of the other players’ behavior and acts rationally.” - Osborne

An action vector from which no player can profitably unilaterally deviate.

Definition

An action tuple \( a \) is a NE if for every \( i \in N \) \( u_i(a, a_i) \geq u_i(b, a_i) \) for all \( b \in A_i \).

Note showing that a point is a NE says nothing about the process by which the steady state is reached. Nor anything about its uniqueness.
Also note that we are implicitly assuming that only pure strategies are possible in this case.

Examples

- Cognitive Radios’ Dilemma
  - Two radios have two signals to choose between \( \{n, w\} \) and \( \{N, W\} \)
  - \( n \) and \( N \) do not overlap
  - Higher throughput from operating as a high power wideband signal when other is narrowband

- Jamming Avoidance
  - Two channels
  - No NE

How do the players find the Nash Equilibrium?

- Preplay Communication
  - Before the game, discuss their options. Note only NE are suitable candidates for coordination as one player could profitably violate any agreement.
- Rational Introspection
  - Based on what each player knows about the other players, reason what the other players would do in its own best interest.
(Best Response - tomorrow) Points where everyone would be playing “correctly” are the NE.
- Focal Point
  - Some distinguishing characteristic of the tuple causes it to stand out. The NE stands out because it’s every player’s best response.
- Trial and Error
  - Starting on some tuple which is not a NE a player “discovers” that deviating improves its payoff. This continues until no player can improve by deviating. Only guaranteed to work for Potential Games (couple weeks)

Nash Equilibrium as a Fixed Point

- Individual Best Response
  \( \hat{B}(a) = \{b \in A : u_i(b, a_i) \geq u_i(a, a_i) \forall a \in A\} \)
- Synchronous Best Response
  \( \hat{B}(a) = \times \hat{B}(a) \)
- Nash Equilibrium as a fixed point
  \( a^* = \hat{B}(a^*) \)
- Fixed point theorems can be used to establish existence of NE (see dissertation)
- NE can be solved by implied system of equations
Example solution for Fixed Point by Solving for Best Response Fixed Point

• Bandwidth Allocation Game
  – Five cognitive radios with each radio, \(i\), free to determine the number of simultaneous frequency hopping channels the radio implements, \(c_i \in [0, \infty)\).
  – Goal: \(u_i(c) = P(c) - C(c)\)
  – \(P(c)\) fraction of symbols that are not interfered with
  – \(C(c)\) is radio \(i\)'s cost for supporting \(c_i\) simultaneous channels.

\[
u_i(c) = B - \sum_{k \in N} c_k - Kc_i\]

Best Response Analysis

Goal:

\[
u_i(c) = \left[ B - \sum_{k \in N} c_k \right] c_i - Kc_i\]

Best Response:

\[c_i = \hat{B}_i(c) = \left( B - K - \sum_{k \in N} c_k \right) / 2\]

Simultaneous System of Equations

\[
\begin{align*}
c_1 & = 0 \cdot c_2 + 0.5 \cdot c_3 + 0.5 \cdot c_4 + 0.5 \cdot c_5 = (B-K)/2 \\
c_2 & = 0.5 \cdot c_1 + 0.5 \cdot c_3 + 0.5 \cdot c_4 + 0.5 \cdot c_5 = (B-K)/2 \\
c_3 & = 0.5 \cdot c_1 + 0.5 \cdot c_2 + 0.5 \cdot c_4 + c_5 = (B-K)/2 \\
c_4 & = 0.5 \cdot c_1 + 0.5 \cdot c_2 + 0.5 \cdot c_3 + 0.5 \cdot c_5 = (B-K)/2 \\
c_5 & = 0.5 \cdot c_1 + 0.5 \cdot c_2 + 0.5 \cdot c_3 + 0.5 \cdot c_4 = (B-K)/2 \\
\end{align*}
\]

Solution:

\[
\hat{c}_i = (B - K) / 6 \quad \forall i \in N
\]

Generalization:

\[
\hat{c}_i = (B - K) / \left( |N| + 1 \right) \quad \forall i \in N
\]

Significance of NE for CRNs

Theorem 4.1: NE and Cognitive Radio Network Steady States (*)
Given cognitive radio network \((N, A, [a_i], [\delta_c], T)\) where all players are autonomous rational, if the game \((N, A, [a_i])\) has an NE \(\alpha^*\), then \(\alpha^*\) is a fixed point for \(T\).

Proof: Suppose \(\alpha^*\) is not a fixed point. Then for some \(i \in N\), there must be some \(\bar{a}_i \in a_i\) with \(\bar{a}_i \neq a^*_i\) such that \(u_i(\bar{a}_i, a^*_{-i}) > u_i(a^*_i, a^*_{-i})\). But this contradicts the assumption that \(\alpha^*\) is an NE. Therefore, \(\alpha^*\) must be a fixed point for \(T\).

• Why not “if and only if”?
  - Consider a self-motivated game with a local maximum and a hill-climbing algorithm.
  - For many decision rules, NE do capture all fixed points (see dissertation)
  - Identifies steady-states for all “intelligent” decision rules with the same goal.
  - Implies a mechanism for policy design while accommodating differing implementations
  - Verify goals result in desired performance
  - Verify radios act intelligently

My Favorite Mixed Strategy Story

Pure Strategies in an Extended Game
Consider an extensive form game where each stage is a strategic form game and the action space remains the same at each stage. Before play begins, each player chooses a probabilistic strategy that assigns a probability to each action in his action set. At each stage, the player chooses an action from his action set according to the probabilities he assigned before play began.

Example
Consider a video football game which will be simulated. Before the game begins, two players assign probabilities of calling running plays or passing plays for both offense or defense. In the simulation, for each down the kind of play chosen by each team is based on the initial probabilities assigned to kinds of plays. (Play NCAA2004)
Example Mixed Strategy Game

<table>
<thead>
<tr>
<th>Jamming game</th>
<th>Action Tuple Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>q</td>
<td>(1-q)</td>
</tr>
<tr>
<td>a₁</td>
<td>p₁a₁</td>
</tr>
<tr>
<td>a₂</td>
<td>p₁a₂</td>
</tr>
<tr>
<td>b₁</td>
<td>(1-p)b₁</td>
</tr>
<tr>
<td>b₂</td>
<td>(1-p)b₂</td>
</tr>
</tbody>
</table>

Best Response Correspondences

Expected Utilities

Sets of probability distributions

Δ(Aₗ)=pₗ(1-pₗ): ∀pₗ∈[0,1]

Δ(Aᵢ)=qᵢ(1-qᵢ): ∀qᵢ∈[0,1]

Nash Equilibrium in a Mixed Strategy Game

Definition Mixed Strategy Nash Equilibrium

A mixed strategy profile α* is a NE iff ∀i∈N

Uᵢ(α*,αᵢ)≥Uᵢ(β,αᵢ)∀β∈Δ(Aᵢ)

Best Response Correspondence

BR(αᵢ*)=arg max Uᵢ(αᵢ,αᵢ)

Alternate NE Definition

Consider B(a)=xᵢ∈BR(a)

A mixed strategy profile α* is a NE iff

αᵢ∈B(α*)

Interesting Properties of Mixed Strategy Games

1. Every Mixed Extension of a Strategic Game has an NE.
2. A mixed strategy αᵢ is a best response to

αᵢ iff every action in the support of αᵢ itself is

a best response to αᵢ.
3. Every action in the support of any player’s equilibrium mixed strategy yields the same payoff to that player.

The Core (Transferrable)

- For (N,v), the set of feasible payoff profiles, (k) for which there is no coalition S and S-

feasible payoff vector (y) for which yᵢ > xᵢ for all i ∈ S.
- General principles of the NE also apply to the Core:
- Number of solutions for a game may be anywhere from 0 to ∞
- May be stable or unstable.

Coalitional Game (with transferable payoff)

- Concept: groups of players called coalitions conspire together to implement actions which yields a result for the coalition. The value received by the coalition is then distributed among the coalition members.
- Where do radios collaborate and distribute value?
  - 802.16h interference groups – allocation of bandwidth
  - File sharing in P2P network
- Where do radios collaborate and distribute value?
  - Distribution of frequencies/spreading codes among cells
- Transferable utility refers to existence of some commodity for which a player’s utility increases by one unit for every unit of the commodity it receives.

Game Components (N,v):

- N set of players
- Characteristic function
- Coalition, S⊆N
- How is this value distributed?
  - Payoff vector, (xᵢ)ᵢ∈N
- Payoff vector is said to be S-feasible if x(S)≥v(S)

Cognitive Radio Technologies, 2007
Example

• Suppose three radios, $N = \{1, 2, 3\}$, can choose to participate in a peer-to-peer network.
• Characteristic Function
  $v(N) = 1$
  $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = \alpha \in [0, 1]$
  $v(1) = v(2) = v(3) = 0$
• Loosely, $\alpha$ indicates # of duplicated files
• If $\alpha > 2/3$, Core is empty

Example adaptations for
\begin{align*}
\alpha = 4/5 & \quad x = (2/5, 2/5, 0) \\
\alpha = 3/5 & \quad x = (0, 3/5, 1/5) \\
\alpha = 1/3 & \quad x = (2/5, 0, 2/5) \\
\end{align*}

Comments on the Core
• Possibility of empty core implies that even when radios can freely negotiate and form arbitrary coalitions, no steady-state may exist
• Frequently very large (infinite) number of steady-states, e.g., $\alpha < 2/3$
  - Makes it impossible to predict exact behavior
• Existence conditions for the Core, but would need to cover some linear programming concepts
• Related (but not addressed today) concepts:
  - Bargaining Sets, Kernel, Nucleolus

Strong NE
• Concept: Assume radios are able to collaborate, but utilities aren’t necessarily transferrable
• An action tuple $a^*$ such that
  $u_i(a_j^*) \geq u_i(a_{-j} \cup i) \forall S \subseteq N, a_j \in \times A_j$

<table>
<thead>
<tr>
<th>No Strong NE</th>
<th>Unique Strong NE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$N$</td>
</tr>
<tr>
<td>(9, 6, 9, 6)</td>
<td>(3, 2, 1)</td>
</tr>
<tr>
<td>$w$</td>
<td>$N$</td>
</tr>
<tr>
<td>(21, 3, 2)</td>
<td>(7, 7)</td>
</tr>
</tbody>
</table>

Motivation for Shapley value
• Core was generally either empty or very large. Want a “good” single solution.
• Terminology
  - Marginal Contribution of $i$
    $\Delta_i(S) = v(S \cup i) - v(S)$
  - Interchangeability of $i, j$
    $\Delta_i(S) = \Delta_j(S) \forall S \subseteq N \setminus \{i, j\}$
  - Dummy player (no synergy)
    $\Delta_i(S) = v(\{i\}) \forall S \subseteq N \setminus i$

Axioms for Shapley Value
• Let $\psi$ be some distribution of value for a TU coalition game
• Symmetry:
  - If $i$ and $j$ are interchangeable, then $\psi(v) = \psi(v)$
• Dummy:
  - If $i$ is a dummy, then $\psi(v) = v(\emptyset)$
• Additivity:
  - Given $(N,v)$ and $(N,w)$, $\psi(v + w) = \psi(v) + \psi(w)$ for all $i \in N$, where $v + w = v(S) + w(S)$
• Balanced Contributions
  - Given $(N,v)$, $\psi_i(N, v) = \psi_i(N \setminus j, v_{\hat{j}})$

Shapley Value
$$\psi_i(S) = \sum_{S \subseteq N, i \in S} \frac{|S||N - S| - 1}{|N!|} (v(S) - v(S \setminus i))$$
Implications of Shapley Value

- One form of a fair allocation
  - What you receive is based on the value you add
  - Independent of order of arrival
  - I liken it to setting salaries according to the Value Over Replacement Player concept in Baseball Statistics

- “Better” solution concept than the core as it’s a single payoff as opposed to a potentially infinite number
- Allows for analysis of relative “power” of different players in the system

Bargaining Problem

- Components: \( \langle F, v \rangle \)
  - Feasible payoffs \( F \), closed convex subset of \( \mathbb{R}^n \)
  - Disagreement Point \( v = (v_1, v_2) \)
- What 1 or 2 could achieve without bargaining

- Example:
  - Even if system is jammed, still gets some throughput
  - Member of 802.16h interference group and try its luck

- \( F \) is said to be essential if there is some \( y \in F \) such that
  \[ y_1 > v_1 \quad \text{and} \quad y_2 > v_2 \]

- If contracts are “binding” then \( F \) could be the payoffs corresponding to entire original action space
- Otherwise, \( F \) may need to be drawn from the set of NE or from enforceable set (see punishment in repeated games)

A particular solution is referred to by \( \phi(F, v) \in \mathbb{R}^n \)

Desirable Bargaining Axioms about a Solution

- Strong Efficiency
  - \( \phi(F, v) \) is Pareto Efficient
- Individually Rational
  - \( \phi(F, v) \geq v \)
- Scale Covariance
  - For any \( \lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathbb{R}, \lambda_1, \lambda_2 > 0 \), if
    \[ G = \{ (\lambda \cdot x + \gamma_1, \lambda \cdot x + \gamma_2) | (x, x) \in F \} \]
    then
    \[ \phi(G, w) = (\lambda \cdot \phi(F, v) + \gamma_1, \lambda \cdot \phi(F, v) + \gamma_2) \]

- Independence of Irrelevant Alternatives
  - If \( G \subseteq F \) and \( G \) is closed and convex and \( \phi(F, v) \in G \), then \( \phi(G, v) = \lambda \cdot \phi(F, v) \)
- Symmetry
  - If \( v_1 = v_2 \) and \( \{(x_1, x_2) | (x_2, x_1) \in F \} = F \), then \( \phi_1(F, v) = \phi_2(F, v) \)

Nash Bargaining Solution

- \( \text{NBS} \) is a solution which simultaneously satisfies the preceding 5 axioms

Centralized Optimization Problem

- \( \text{Max}_{x \in \mathbb{R}^n} \prod_{i=1}^{N} (x_i - MR_i) \)
  \[ \text{st.} \quad \begin{cases} x_i \geq MR, & i \in [1..N] \\ x_i \leq PR, & i \in [1..N] \\ (Ax)_l \geq (C), & l \in [1..L] \end{cases} \]
- Unique NBS exists

GT framework for BW allocation [Yaiche]: System Model

- \( N \) users
- \( L \) links
- Users compete for the total link capacity
- Each user has a minimum rate \( MR_i \) and peak rate \( PR_i \)
- Admissible rate vector is given by:
  \[ X_0 = \{ x \in \mathbb{R}^n \mid x \geq MR, x \leq PR, \text{ and } Ax \leq C \} \]
- \( C \) : vector of link capacities
- \( A \in \mathbb{R}^{m \times n}, a_{pq} = 1 \) if link belongs to path \( p \), else 0.

### Steady-State Summary

- Not every game has a steady-state
- NE are analogous to fixed points of self-interested decision processes
- NE can be applied to procedural and ontological radios
  - Don’t need to know decision rule, only goals, actions, and assumption that radios act in their own interest
- A game (network) may have 0, 1, or many steady-states
- All finite normal form games have an NE in its mixed extension
  - Over multiple iterations, implies constant adaptation
- More complex game models yield more complex steady-state concepts
- Can define steady-states concepts for coalitional games
  - Frequently so broad that specific solutions are used