Evaluating Equilibria

Objective Function Maximization, Pareto Efficiency, Notions of Fairness

Optimality

- In general we assume the existence of some design objective function $J:A\to\mathbb{R}$
- The desirableness of a network state, *a*, is the value of *J*(a).
- In general maximizers of *J* are unrelated to fixed points of *d*.

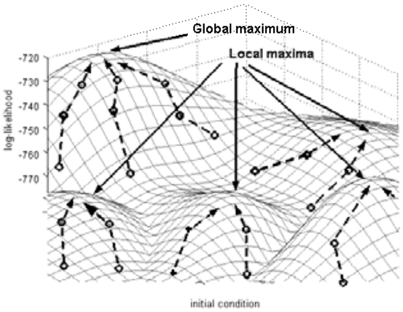
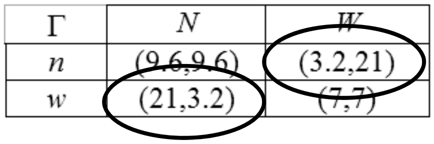


Figure from Fig 2.6 in I. Akbar, “Statistical Analysis of Wireless Systems Using Markov Models,” PhD Dissertation, Virginia Tech, January 2007

Example Functions

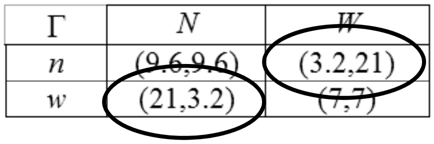
- Utilitarian
  - Sum of all players' utilities
  - Product of all players' utilities
- Practical
  - Total system throughput
  - Average SINR
  - Maximum End-to-End Latency
  - Minimal sum system interference
- Objective can be unrelated to utilities

Utilitarian Maximizers



Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

System Throughput Maximizers



Γ	N	W
n	(9.6,9.6)	(3.2,21)
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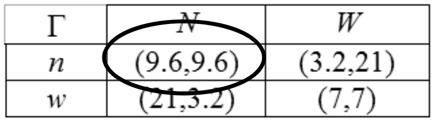
Interference Minimization



Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

Price of Anarchy (Factor)

$$\text{Ⓐ} = \frac{\text{Performance of Centralized Algorithm Solution}}{\text{Performance of Distributed Algorithm Solution}}$$

$$\text{Ⓐ} \geq \mathbf{1}$$

- Centralized solution always at least as good as distributed solution
  - Like ASIC is always at least as good as DSP
- Ignores costs of implementing algorithms
  - Sometimes centralized is infeasible (e.g., routing the Internet)
  - Distributed can sometimes (but not generally) be more costly than centralized

Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

$$\text{Ⓐ} = \frac{9.6}{7}$$

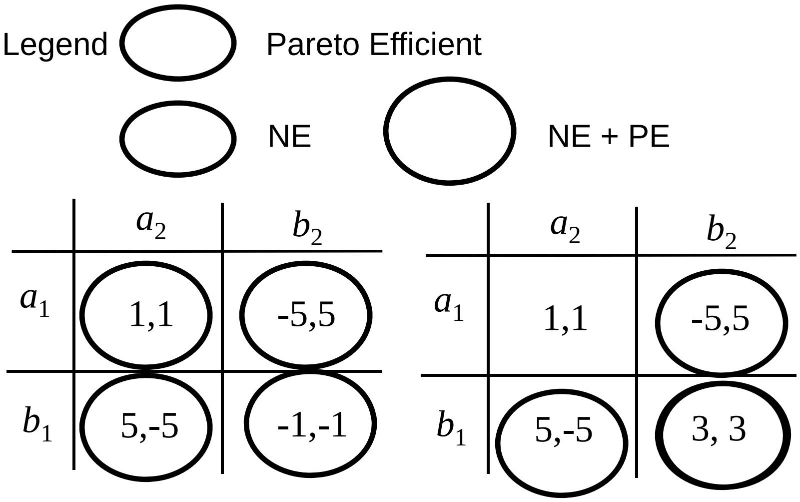
Price of Anarchy Discussion

- Best of All Possible Worlds
  - Low complexity distributed algorithms with low anarchy factors
- Reality implies mix of methods
  - Hodgepodge of mixed solutions
    - Policy – bounds the price of anarchy
    - Utility adjustments – align distributed solution with centralized solution
    - Market methods – sometimes distributed, sometimes centralized
    - Punishment – sometimes centralized, sometimes distributed, sometimes both
    - Radio environment maps –"centralized" information for distributed decision processes
  - Fully distributed
    - Potential game design – really, the Panglossian solution, but only applies to particular problems

Pareto efficiency (optimality)

- **Formal definition**: An action vector *a** is ***Pareto efficient*** if there exists no other action vector *a*, such that every radio's valuation of the network is at least as good and at least one radio assigns a higher valuation
- **Informal definition:** An action tuple is *Pareto efficient* if some radios must be hurt in order to improve the payoff of other radios.
- **Important note**
  - Like design objective function, unrelated to fixed points (NE)
  - Generally less specific than evaluating design objective function

Example Games



	a_2	b_2
a_1	1,1	-5,5
b_1	5,-5	-1,-1

	a_2	b_2
a_1	1,1	-5,5
b_1	5,-5	3, 3

Notions of Fairness

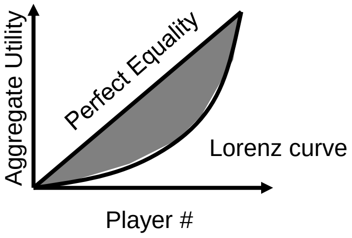
- What is “Fair”?
  - Abstractly “fair” means different things to different analysts
  - In every day life, “unfair” is short hand for “I deserve more than I got”
- Nonetheless is used to evaluate how equitably radio resources are distributed

Gini Coefficient

- Basic concept:
  - Order players by utility.
  - Form CDF for sorted utility distribution (Lorenz curve)
  - Integrate (sum) the difference between perfect equality (of outcome) and CDF
  - Divide result by sum of all players' utilities
- Formula

$$G(a)=\frac{1}{n}\left(n+1-2\frac{\sum_{i\in N}(n+1-i)u_i(a)}{\sum_{i\in N}u_i(a)}\right)$$

- Used in a lot of macro-economic comparisons of income distributions
- Relatively simple, independent of scale, independent of size of *N*, anonymity
- Radically different outcomes can give the same result



Γ	N	W
n	(9.6,9.6)	(3.2,21)
w	(21,3.2)	(7,7)

G	N	W
n	0	0.37
w	0.37	0

Other Metrics of Fairness

- Theill Index

$$T(a)=\frac{1}{n}\sum_{i\in N}\left(\frac{u_i(a)}{\bar{u}}\ln\frac{u_i(a)}{\bar{u}}\right) \qquad \bar{u}(a)=\frac{1}{n}\sum_{i\in N}u_i(a)$$

- Atkinson Index, ε is income inequality aversion

$$T(a)=1-\frac{1}{\bar{u}}\left(\frac{1}{n}\sum_{i\in N}u_i(a)^{1-\varepsilon}\right)^{1/(1-\varepsilon)},\varepsilon\in[0,1)$$

$$T(a)=1-\frac{1}{\bar{u}}\left(\frac{1}{n}\sum_{i\in N}u_i(a)\right)^{1/n},\varepsilon=1$$

Bargaining Problem

- Components: $\langle F, v\rangle$
  - Feasible payoffs F, closed convex subset of $\mathbb{R}^n$
  - Disagreement Point $v = (v_1, v_2)$
    - What 1 or 2 could achieve without bargaining
- Example:
  - Even if system is jammed, still gets some throughput
  - Member of 802.16h interference group and try its luck
- F is said to be *essential* if there is some $y \in F$ such that $y_1 > v_1$ and $y_2 > v_2$
- If contracts are “binding” then F could be the payoffs corresponding to entire original action space
- Otherwise, F may need to be drawn from the set of NE or from enforceable set (see punishment in repeated games)
- A particular solution is referred to by $\phi(F, v) \in \mathbb{R}^n$

Desirable Bargaining Axioms about a Solution

- Strong Efficiency
  - $\phi(F, v)$ is Pareto Efficient
- Individually Rational
  - $\phi(F, v) \geq v$
- Scale Covariance
  - For any $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 > 0$, if

$$G = \left\{ \left(\lambda_1 x_1 + \gamma_1, \lambda_2 x_2 + \gamma_2 \right) | \left(x_1, x_2 \right) \in F \right\}$$

then

$$\phi(G, w) = \left(\lambda_1 \phi_1 \left(F, v \right) + \gamma_1, \lambda_2 \phi_2 \left(F, v \right) + \gamma_2 \right)$$

$$w = \left(\lambda_1 v_1 + \gamma_1, \lambda_2 v_2 + \gamma_2 \right)$$

- Independence of Irrelevant Alternatives
  - If $G \subseteq F$ and G is closed and convex and $\phi(F, v) \in G$, then $\phi(G, v) = \phi(F, v)$
- Symmetry
  - If $v_1 = v_2$ and $\{(x_1, x_2) | (x_2, x_1) \in F\} = F$, then $\phi_1(F, v) = \phi_2(F, v)$

Nash Bargaining Solution

- NBS

$$\phi\left(F,v\right)\in\arg\max_{x\in F,x\geq v}\prod_{i\in N}\left(x_i-v_i\right)$$

- Interestingly, this is the only bargaining solution which simultaneously satisfies the preceding 5 axioms

GT framework for BW allocation [Yaiche]: System Model

- *N* users
- *L* links
- Users compete for the total link capacity
- Each user has a minimum rate MR_i and peak rate PR_i
- Admissible rate vector is given by,

$$X_0=\left\{x\in\mathbb{R}^N \mid x\geq MR,\ x\leq PR,\ \text{and}\ Ax\leq C\right\}$$

C : vector of link capacities

A_{L*N}: a_{lp} = 1 if link belongs to path *p*, else 0.

Scenario given in H. Yaiche, R. Mazumdar, C. Rosenberg, “A game theoretic framework for bandwidth allocation and pricing in broadband networks”, IEEE/ACM Transactions on Networking, Volume: 8 , Issue: 5 , Oct. 2000, pp. 667-678.

Centralized Optimization Problem

- $$\begin{aligned} & Max_{\{x\}} \prod_{i=1}^{N}\left(x_i - MR_i\right) \\ st:\ & \underbrace{x_i \ge MR_i \quad i \in \{1 \ldots N\}}_{v} \\ & \underbrace{\begin{aligned} & x_i \le PR_i \quad i \in \{1 \ldots N\} \\ & \left(Ax\right)_l \ge \left(C\right)_l \quad l \in \{1 \ldots L\} \end{aligned}}_{F} \end{aligned}$$
- Unique NBS exists

Summary of Equilibria Evaluation

- Lots of different ways which a point can be evaluated
- Many are contradictory
- Loosely, any point could be said to be optimal given the right objective function
- Insufficient to say that a point is optimal
  - Must describe the metric in use
- Suggestion: use whatever metric makes sense to you as a network designer